MA114 Summer 2018
Worksheet 27 – Separable Differential Equations
7/31/18

1. Use separation of variables to find the general solutions to the following differential equations.

   a) $y' + 4xy^2 = 0$
   b) $\sqrt{1-x^2}y' = xy$
   c) $(1+x^2)y' = x^3y$
   d) $\sqrt{1+y^2}y' + \sec(x) = 0.$

2. Solve the initial value problems.

   a) $\frac{dL}{dt} = kL^2 \ln t, L(1) = -1$, where k is a constant.
   b) $\frac{dP}{dt} = \sqrt{Pt}, P(1) = 2$

1a) $\frac{dy}{dx} = -4xy^2$

$\int \frac{dy}{y^2} = \int -4x\,dx$

$-\frac{1}{y} = -2x^2 + C$

$\boxed{y = \frac{1}{2x^2+C}}$

1b) $\frac{dy}{dx}\sqrt{1-x^2} = xy$

$\int \frac{1}{y}\,dy = \int \frac{x}{\sqrt{1-x^2}}\,dx$ $\quad u = 1-x^2$

$\ln|y| = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$

$= -\sqrt{1-x^2} + C$

$|y| = e^{-\sqrt{1-x^2}}e^C$

$\boxed{y = Ae^{-\sqrt{1-x^2}}}$

1c) $(1+x^2)y' = x^3y$

$\int \frac{1}{y}\,dy = \int \frac{x^3}{1+x^2}\,dx$

$$\begin{array}{r} x \\ 1+x^2 \overline{)\,x^3 \quad} \\ -(x^3+x) \\ \hline -x \end{array}$$

$\ln|y| = \int x - \frac{x}{1+x^2}\,dx$

$= \frac{1}{2}x^2 - \frac{1}{2}\ln|1+x^2| + C$

$|y| = e^{\frac{1}{2}x^2} e^{\ln((1+x^2)^{-\frac{1}{2}})} e^C$

$\boxed{y = \frac{Ae^{\frac{1}{2}x^2}}{\sqrt{1+x^2}}}$

1d) $\sqrt{1+y^2}\,y' + \sec(x) = 0$

$\int \sqrt{1+y^2}\,dy = \int -\sec(x)\,dx$

$y = \tan\theta$
$dy = \sec^2\theta\,d\theta$

$\int \sec^3\theta\,d\theta = -\ln|\sec(x)+\tan(x)| + C$

$\frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|) = -\ln|\sec(x)+\tan(x)| + C$

$\boxed{\frac{1}{2}\left(y\sqrt{1+y^2} + \ln|\sqrt{1+y^2}+y|\right) = -\ln|\sec(x)+\tan(x)| + C}$

2) a) $\frac{dL}{dt} = kL^2 \ln(t)$, $L(1) = -1$

$$\int \frac{dL}{L^2} = \int k \ln(t)\, dt = k\left(t \ln(t) - \int dt\right)$$

$$-\frac{1}{L} = k(t \ln(t) - t) + c$$

$$L = -\frac{1}{k(t(\ln(t) - t)) + c}$$

Apply I.C: $-1 = k(0 - 1) + c$

$$c = k - 1$$

$$L = -\frac{1}{k(t\ln(t) - t) + k - 1}$$

b) $\frac{dP}{dt} = \sqrt{Pt}$, $P(1) = 2$

$$\int \frac{dP}{\sqrt{P}} = \int \sqrt{t}\, dt$$

$$2\sqrt{P} = \frac{2}{3}t^{3/2} + C$$

$$\sqrt{P} = \frac{1}{3}t^{3/2} + C$$

$$P = \left(\frac{1}{3}t^{3/2} + c\right)^2$$

Apply I.C: $2 = \left(\frac{1}{3}(1)^{3/2} + c\right)^2$

$$\sqrt{2} = \frac{1}{3} + C$$

$$\boxed{C = \sqrt{2} - \frac{1}{3}}$$

$$\boxed{P = \left(\frac{1}{3}t^{3/2} + \sqrt{2} - \frac{1}{3}\right)^2}$$